Devoir Maison: "Images et géométrie discrète"

Introduction

The objective of this project is to designed numerical descriptors for shape retrieval from a database. The project is thus composed of two parts: during the first exercise, we first focus on global estimators and their convergence properties. The second one describes the shape retrieval process and details its specifications.

We expect from you:

- A short report with answers to the "formal" questions and description of the your implementations choices.
- A C++ project (`CMakeLists.txt` plus several **commented** cpp program files).

All materials can be obtained in the following adresses (illustrated with command-line SVN commandes):

```
%% This subject, image database for section 2
svn checkout https://svn.liris.cnrs.fr/dcoeurjo/ENS-TP/DM/

%% DGtal examples
svn checkout https://svn.liris.cnrs.fr/dcoeurjo/ENS-TP/DGtalSkel/
```

1 Multigrid analysis of geometrical moments

"Petit petit, ça devient plus petit"

1.1 Properties of moments

We are interest in the analysis of geometrical moment of order $m_{p,q}$ defined as follows for $X \subset \mathbb{R}^2$:

$$m_{p,q}(X) = \int\int_X x^p y^q \, dxdy$$

On a digital object $Z \in \mathbb{Z}^2$, we will use the following approximation:

$$\hat{m}_{p,q}(Z) = \sum_{(i,j)\in Z} i^p j^q$$

Similarly, we will also consider **central geometrical moments** defined by:

$$\mu_{p,q}(X) = \int\int_X (x-\mu_x)^p (y-\mu_y)^q \, dxdy$$

$$\hat{\mu}_{p,q}(Z) = \sum_{(i,j)\in Z} (i-\mu_i)^p (p-\mu_j)^q$$

where (μ_x, μ_y) (resp. (μ_i, μ_j)) is the centroid of X (resp. Z).

Question 1 *Show that $\mu_{p,q}$ and $\hat{\mu}_{p,q}$ are translation invariant.*

Question 2 *Express* (μ_x, μ_y) *coordinates as function of* $m_{a,b}$ *for some* $a, b \in \mathbb{Z}$

Question 3 *Express first* $\mu_{p,q}$ *central moments* ($p + q \leq 2$) *as functions of* $m_{p,q}$.

Question 4 *Geometrical moments are not scale-invariant. We consider a scaling* X *by a factor* k *(denoted* $k \cdot X$*), express* $m_{p,q}(k \cdot X)$ *as a function of* $m_{p,q}(X)$

Similarly, you also have $\mu_{p,q}(k \cdot X)$ as a function of $\mu_{p,q}(X)$. We can use the previous result to design moments $\eta_{p,q}$ with scale invariant properties.

Question 5

- *Express* $m_{0,0}(k \cdot X)$ *from* $m_{0,0}(X)$.
- *Find the* α *such that*
$$\frac{m_{p,q}(k \cdot X)}{m_{0,0}(k \cdot X)^\alpha} = \frac{m_{p,q}(X)}{m_{0,0}(X)^\alpha}$$
- *Define* $\eta_{p,q}$ *as a function of* $\mu_{p,q}$ *and* $\mu_{0,0}$. *Conclude on the fact that* $\eta_{p,q}$ *are translation and scale invariant.*

We are now looking for rotational invariant moments. If we consider a rotation of X to X' by angle θ:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We can get:

$$m_{20}(X') = \left(\frac{1+\cos\theta}{2}\right) m_{20}(X) - (\sin 2\theta) m_{11}(X) + \left(\frac{1-\cos\theta}{2}\right) m_0 2(X) \quad (1)$$
$$m_{11}(X') = \left(\frac{\sin 2\theta}{2}\right) m_{20}(X) + (\cos 2\theta) m_{11}(X) - \left(\frac{\sin 2\theta}{2}\right) m_0 2(X) \quad (2)$$
$$m_{02}(X') = \left(\frac{1-\cos\theta}{2}\right) m_{20}(X) + (\sin 2\theta) m_{11}(X) + \left(\frac{1+\cos\theta}{2}\right) m_0 2(X) \quad (3)$$

Question 6 *Demonstrate that* $\phi_1 = m_{20} + m_{02}$ *is rotation invariant (*i.e. *prove that* $\phi_1(X) = \phi_1(X')$*).*

Question 7 *Demonstrate that* $\phi_2 = (m_{20} - m_{02})^2 + 4m_{11}$ *is rotation invariant.*

To design translation, scale and rotational invariant moments from $m_{p,q}$, we just have to merge all previous results and define

$$\phi_1 = \eta_{20} + \eta_{02} \quad (4)$$
$$\phi_2 = (\eta_{20} - \eta_{02})^2 + 4\eta_{11} \quad (5)$$
$$\phi_3 = \eta_{20}\eta_{02} - \eta_{11}^2 \quad (6)$$
$$\phi_4 = (\eta_{30} - 3\eta_{12})^2 + (3\eta_{21} - \eta_{03}) \quad (7)$$
$$\phi_5 = (\eta_{30} + \eta_{12})^2 + (\eta_{21} + \eta_{03})^2 \quad (8)$$

A complete theory of affine invariant moment exists. For example, Legendre moment theory provides a way to define high order invariant moments as linear combination of geometrical moments. The key point here is to observe that all discussed moments $(\mu, \eta, \phi, ..)$ are functions of m_{pq}.

1.2 Algorithmic details

The overall objective of this section is to define and implement geometrical moment estimators, and to evaluate the convergence speed of moments $m_{p,q}$.

Question 8 *Implement a naive version of the moment m_{pq} computation: Given a digital set and p and q parameters, return the sum of $x^p y^q$ for each point of the digital set. What is the computational cost (detail your computation model) ?*

To optimize the process, we will use additivity property of moments:

Question 9 *Show that for two disjoint sets $A, B \subset \mathbb{R}^2$, $m_{pq}(A \cup B) = m_{pq}(A) + m_{pq}(B)$. Similarly, if $A \subset B$, $m_{pq}(B \setminus A) = m_{pq}(B) - m_{pq}(A)$.*

We suppose that the digital set Z is connected without hole.

Question 10 *Give the explicit formula to compute the geometrical moment m_{pq} of a vertical set of points B_{ij}: $(i, 0) \to (i, j)$. More formally, $B_{ij} = \{(i, y) \mid 0 \leq y \leq j\}$. From such $B_{..}$ sets, and by additivity, compute the geometrical moment m_{pq} of the rectangle $\{(x, y), (x, y')\}$.*

Question 11 *Using the fact that Z is hole-free, and using the border tracker algorithm you have implemented in previous TP, design and implement a more efficient algorithm to compute $m_{pq}(Z)$*

1.3 Multigrid Analysis

Here, we evaluate the multigrid convergence of $\hat{m}_{pq}$ estimators. First, remember that we consider here Gauss digitization at gridstep h of $X \subset \mathbb{R}^2$:

$$Z = Dig(X, h) = \left(\frac{1}{h} \cdot X\right) \cap \mathbb{Z}^2 = X \cap (h \cdot \mathbb{Z}^2)$$

As discussed above, we denote

$$\hat{m}_{pq}(Z) = \sum_{(i,j) \in Z} i^p j^q$$

and define

$$\hat{m}_{pq}(Z, h) = \hat{m}_{pq}(h \cdot Z)$$

Question 12 *From the definition and using the result of Question 4, express $\hat{m}_{pq}(Z, h)$ as a function of $\hat{m}_{pq}(Z)$ and h.*

Question 13 *Perform a multigrid analysis of $\hat{m}_{pq}(Dig(X, h), h)$*

- *Implement function which constructs the digitization of an Euclidean ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at grid step h (you may already have one for Euclidean discs from previous TP)*
- *For the first moments ($p+q \leq 3$), output $|\hat{m}_{pq}(Dig(X, h), h) - m_{pq}(E)|$ values for h tending to 0.*
- *Plot error graphs $h \times Error$ in logscale using `gnuplot`*

See Appendix A for first geometrical moments of the ellipse.

Question 14 *We guess that $\hat{m}_{pq}$ has an error in $O(h^{\alpha_{pq}})$ for some $\alpha_{pq} \in \mathbb{R}$ depending only on p and q. Experimentally, can you estimate such α_{pq} for first moments ? (Hint: the slope β of linear fitting in logscale gives you the exponent of something on x^β). Can you guess the general form for the error of m_{pq}.*

So far, we saw in the lectures that $\hat{m}_{00}$ is convergent with speed at least $O(h)$ ([Gauss] with general convex hypothesis on X). Hence, we have $\alpha_{00} = 1$. Adding hypothesis on ∂X (e.g. being C^3), we have $\alpha_{00} = \frac{15}{11} - \epsilon$ [Huxley]

2 Shape Retrieval

"Ceci n'est pas une pipe"

The challenge here is to design shape indexing and shape retrieval processes for the database of shapes in the `original/` folder. More precisely, you'll find in this folder a bunch of PGM binary objects. For example, you have 20 different hammers. The problem consists in the following process:

- Shape indexing: for each shape, associate a small vector $\vec{v} \in \mathbb{R}^d$ of d scalar quantities describing the shape (for instance, invariant geometrical moments...). We suppose that each shape of the database has got such scalar description.
- Shape retrieval: given a input shape, we compute its feature vector $\vec{w}$ and we output for instance the $k-$nearest shape for a given similarity metric on vectors $d(\vec{v}, \vec{w})$ (which could simply be the l_2 norm iun $\mathbb{R}^2$ but could also be more complex).

Usually, we would add a data-mining and classification tools based on these information to finally output that the given shape seems to be a hammer with probability 0.95.

We would like you to design and implement such tools. You can use any descriptor you want (moments, contour based curvature, ...) and any metric you prefer. We just would like to see for instance the 20 nearest shape from the database. Note the shapes are split into classes defined in the filenames. For instance, in the 20 nearest shapes, we could expect to retrieve most of the shapes belonging to the same class as the input one. You could for example compute the score of the request as the amount of shapes from the same class retrieved in the first 20 ones.

In `DGtalSkel`, you will find an example (`PGMReader.cpp`) to load a PGM file with DGtal tools and to convert pixels with intensity greater than 0 into a digital set.

A Geometrical moments of a General Ellipse

Notations:

- a length of the semi-major axis
- b length of the semi-minor axis
- x_0, y_0 coordinate of the center of the ellipse
- λ angle of the major axis with the $x-$axis

$$m_{00} = \pi ab \tag{9}$$

$$m_{10} = \pi ab x_0 \tag{10}$$

$$m_{01} = \pi ab y_0 \tag{11}$$

$$m_{20} = \pi ab \left(\frac{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}{4} + x_0^2 \right) \tag{12}$$

$$m_{02} = \pi ab \left(\frac{a^2 \sin^2 \lambda + b^2 \cos^2 \lambda}{4} + y_0^2 \right) \tag{13}$$

$$m_{11} = \pi ab \left(\frac{(a^2 - b^2) \cos \lambda \sin \lambda}{4} + x_0 y_0 \right) \tag{14}$$

$$m_{30} = \pi ab \left(\frac{3x_0(a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)}{4} + x_0^3 \right) \tag{15}$$

$$m_{03} = \pi ab \left(\frac{3y_0(a^2 \sin^2 \lambda + b^2 \cos^2 \lambda)}{4} + y_0^3 \right) \tag{16}$$

$$m_{21} = \pi ab \left(\frac{y_0(a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)}{4} + \frac{x_0(a^2 - b^2) \sin \lambda \cos \lambda}{2} + x_0^2 y_0 \right) \tag{17}$$

$$m_{12} = \pi ab \left(\frac{x_0(a^2 \sin^2 \lambda + b^2 \cos^2 \lambda)}{4} + \frac{y_0(a^2 - b^2) \sin \lambda \cos \lambda}{2} + x_0 y_0^2 \right) \tag{18}$$