Assignment: “Images et géométrie discrète”

Introduction

The objective of this project is to perform experimental evaluation of multigrid behavior of differential estimators. This project is based on practical-work TP6 and TP7.

We expect from you:

- A short report with answers to the ”formal” questions and description of the your implementations choices (Sect. 2)
- A C++ project (`CMakeLists.txt` plus couple of **commented** cpp program files).

1 Multigrid analysis of Maximal Segments

We would like to evaluate experimentally the behavior of maximal segments in a multigrid framework. More precisely, we would like to evaluate the following quantities:

- The number of maximal segments.
- The max/min values of maximal segment lengths.
- The average length of maximal segments.

More precisely, we can to evaluate the following statement:

Lemma 1 (Asymptotic Laws of Maximal Segments) *Let X be some convex shape of $\mathbb{R}^2$, with at least C^3-boundary and bounded curvature. The discrete length (number of points) of maximal segments in ∂Z for $Z = Dig(X, h)$ follows:*

- *the shortest is lower bounded by $\Omega(h^{-\frac{1}{3}})$;*
- *the longest is upper bounded by $O(h^{-\frac{1}{2}})$;*
- *their average length, denoted $L_D(Z)$, is such that:*

$$\Theta(h^{-\frac{1}{3}}) \leq L_D(Z) \leq \Theta(h^{-\frac{1}{3}} \log\left(\frac{1}{h}\right)) \,. \quad (1)$$

Question 1 *Implements a piece of code that first consider multigrid digitization of an Euclidean convex object (e.g. sphere or ellipse from implicit equation) and extract its contour. Perform a complete maximal covering of the contour into maximal DSS.*

Question 2 *Perform a complete multigrid analysis to verify Lemma 1: statistics of the maximal segment length distribution, behavior (graphs) when h tends to 0...*

Question 3 *Can you conclude something on the probability, for spheres or ellipses, that maximal segment have pathological $O(h^{-\frac{1}{2}})$ length cases ? For a fixed resolution, where those pathological cases are located on the contour ?*

Question 4 *Consider now digitization of implicit non-convex shapes (please consider a non-convex shape with various inflection points). Is Lemma 1 still valid ?*

2 Extended Euclid's Algorithm

Let us consider the following Euclidean division algorithm.

Procedure Convergents((a,b), (p,q), (p',q'), i)

Input: (a, b), (p, q), (p', q'), i
Output: (p', q')
Let r be the remainder of the Euclidean division b/a;
Let u be the quotient of the Euclidean division b/a;
$p'' \leftarrow up' + p$;
$q'' \leftarrow uq' + q$;
if $r > 0$ **then**
  return *Convergents*((r, a), (p', q'), $(p'', q''), i + 1$);
else
  return (p', q')

Question 5 *Let* $(p_{-1}, q_{-1}) = (1, 0)$ *and* $(p_0, q_0) = (0, 1)$*, what is the output of* `Convergents`*(*$(5, 8)$*,* (p_{-1}, q_{-1})*,* (p_0, q_0)*, 0) ?*

Question 6 *Let us consider* `Convergents`*(*(a, b)*,* (p_{-1}, q_{-1})*,* (p_0, q_0)*, 0) (with* $0 \leq a < b$ *and* $\gcd(a, b) = 1$*). We index the recursive calls by* $i = 1 \ldots n$*. Show that*

$$\forall i = 1 \ldots n\,, p_i = u_i p_{i-1} + p_{i-2} \text{ and } q_i = u_i q_{i-1} + q_{i-2}\,.$$

Question 7 *Similarly, with* $r_{-1} = b$ *and* $r_0 = a$*, show that*

$$\forall i = 1 \ldots n\,, r_i = r_{i-2} - u_i r_{i-1}$$

Question 8 *Following previous results, prove the following statements:*

1. $\forall i = 1 \ldots n\,, p_{i-1}q_i - q_{i-1}p_i = \pm 1$
2. $\forall i = -1 \ldots n\,, p_i b - q_i a = \pm r_i$

Since $r_n = gcd(a, b) = 1$*, what is* $p_n b - q_n a$ *?*

Question 9 *Give the definition of uni-modularity. What is the geometrical interpretation of this definition ?*

Question 10 *In the domain in appendix*[1]*, draw the Euclidean segment* $[(0, 0) - (b, a)]$ *and all convergents* (q_i, p_i) *for the input given in Question 5. With respect to the parity of* i*, can you say something on the position of convergents with respect to the segment ? If we construct a polygonal curve with only convergents* (q_i, p_i) *with even index* i *(plus a last point* (b, a)*). What kind of geometrical object I have constructed ?*

Question 11 *Let* L_{odd} *(resp.* L_{even}*) be the polygonal curve of convergents with odd index (resp. even index). Furthermore, we add the point* (b, a) *to the end of each list. For the input given in Question 5, are there integer points between* L_{odd} *and* L_{even} *? Why ?*

Question 12 *For a general setting* `Convergents`*(*(a, b)*,* (p_{-1}, q_{-1})*,* (p_0, q_0)*, 0), can you prove the statement of the previous question ?*

Question 13 *What is the complexity of* `Convergents` *with respect to* a *and* b *?*

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[1] Please checkout the git project to get the PDF file of the figure.